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Scaling laws and vortex profiles in two-dimensional decaying turbulence

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We use high resolution numerical simulations over several hundred of turnover times to study the influence of small scale dissipation onto vortex statistics in 2D decaying turbulence. A scaling regime is detected when the scaling laws are expressed in units of mean vorticity and integral scale, like predicted in Carnevale *et al.*, Phys. Rev. Lett. **66**, 2735 (1991), and it is observed that viscous effects spoil this scaling regime. The exponent controlling the decay of the number of vortices shows some trends toward $\xi = 1$, in agreement with a recent theory based on the Kirchhoff model [C. Sire and P. H. Chavanis, Phys. Rev. E **61**, 6644 (2000)]. In terms of scaled variables, the vortices have a similar profile with a functional form related to the Fermi-Dirac distribution.

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In recent years, two-dimensional turbulence has received rather large interest because of its applications in astrophysics and geophysics and its relative accessibility to numerical simulations with respect to fully developed three-dimensional turbulence. Two-dimensional flows are characterized by the presence of coherent structures (the vortices) which dominate the dynamics. The relaxation of two-dimensional (2D) decaying turbulence is a three-stage process: during an initial transient period, the fluid organizes itself from random fluctuations and a population of coherent vortices emerges. Then, when two like-sign vortices come into contact they merge and form a bigger structure. As time goes on, the vortex number decreases and their average size increases, in a process reminiscent of a coarsening stage. Finally, when only one dipole is left, it decays diffusively due to inherent viscosity.

Two types of studies have been conducted to characterize this relaxation process: some have focused on the precise structure of the vortices (vorticity profiles, ω-ψ relationships, etc.), while others described how the average vortex properties (typical radius, core vorticity, vortex number, etc.) evolve with time. There was some attempt to predict the final state (the dipole) in terms of statistical mechanics of the 2D Euler equation [1]. It is found that a prediction from the initial condition leads to incorrect results due to the effect of viscosity which dissipates the high order moments of the vorticity during the long evolution of the flow towards that state. However, if the constants of the motion are evaluated at later times (i.e., before the last mergings) the prediction improves [2]. This implies that the statistical theory cannot predict the final state of a long viscous evolution but is likely to describe correctly the structure of a vortex that forms after a rapid merging. It was therefore suggested that the isolated vortices of 2D turbulence are sort of local equilibrium states or "maximum entropy bubbles" [3].

Other authors have chosen to disregard the precise structure of the vortices in order to study how the typical characteristics of the flow evolve with time. In such experiments and numerical simulations, it is found that the vortex density *n*, their average radius *a*, and their typical core vorticity ω seem to follow power laws. Two different scenarios have been proposed. In the first one, known as the Batchelor theory [4], the assumption of a unique invariant (the energy $E \sim n\omega^2 a^4$), and the occurrence of a unique relevant time scale $\omega^{-1} \sim t$ implies that the number of vortices *n* decays as $n \sim t^{-\xi}$, with $\xi = 2$. This theory also implies the occurrence of a unique length scale, the typical distance between vortices $n^{-1/2}$, of the same order of magnitude as their typical radius *a*. Hence, the total area occupied by the vortices na^2, or alternatively the Kurtosis $K \sim (na^2)^{-1}$, remains constant, implying $a \sim t$, while the enstrophy $Z \sim n\omega^2 a^2$ decays like $\sim t^{-2}$. However, hyperviscous simulations [5] and experiments [6] suggest a different scenario in which the typical core vorticity ω is an additional invariant. Assuming $n \sim t^{-\xi}$ and the energy conservation such that na^4 is now constant, the scaling theory consistent with this scenario [7] leads to the slow decrease of the total area occupied by the vortices $na^2 \sim t^{-\xi/2}$ (or $a \sim t^{\xi/4}$). The enstrophy now decays as $Z \sim t^{-\xi/2}$ while the Kurtosis increases as $K \sim t^{\xi/2}$. The occurrence of an extra dimensionless relevant parameter na^2 prevents the determination of ξ from purely dimensional grounds as was done within Batchelor theory.

From the numerical and experimental side, the situation is rather confusing. Matthaeus *et al.* [8] performed a very long direct numerical simulation (DNS) of the Navier-Stokes equation and found that the enstrophy decays approximately like t^{-1}. More recently, other DNS at very large resolution [9] produced a similar decay rate $Z \sim t^{-0.8}$. By contrast, in numerical simulations using hyperviscosity (HDNS) [10] the enstrophy decays like $Z \sim t^{-0.3}$. These hyperviscous simulations show an overall agreement with the second scenario with an exponent $\xi \sim 0.75$. Recent simulations with very high resolution agree with these results [11].

In a first series of experiments for which 3D effects were not fully controlled, Cardoso *et al.* [12] obtained scaling laws compatible with the first scenario (roughly conserved vortex area coverage), but with $\xi \approx 0.44$ instead of $\xi = 2$. In a second series of experiments with stratification [6], the same group obtained scaling laws in favor of the second scenario with $\xi \approx 0.7$. In both cases, dissipation is provided not via a standard viscosity, but mainly via friction at the bottom of the experimental apparatus. A simple rescaling, however, can make the experimental system equivalent to a real 2D system, with a time dependent viscosity [6].

Theoretical attempts have been made to understand and clarify the decay process. Among them, simple models describing vortex aggregation process have used point vortices following a Kirchhoff-Hamilton dynamics and merging via empirical rules derived from imposed conservation laws. The model corresponding to the second scenario with constant energy and core vorticity was first investigated in Ref. [10], leading to $\xi \sim 0.75$ [10,13], in agreement with HDNS and experiments. Recently, the same model was investigated using a renormalization group procedure which allows for much larger simulation times (three more decades in time) [14]. The true scaling regime is only obtained for times much larger than previous simulation or experimental times, and the asymptotic decay exponent is found to be $\xi \approx 1$. Interestingly, in the time range comparable to that of HDNS and experiments, the function $n(t)$ displays a pseudoscaling range with an effective exponent $\xi \simeq 0.7$. An effective three-body theory shows that the decay of the total area occupied by vortices results in a situation where mergings occur principally via three-body collisions involving vortices of different signs. A kinetic theory based on these three-body processes leads to $\xi = 1$, in agreement with the simulations. However, since the conservation laws are built in the model *a priori*, there is a definite need for more precise comparisons with DNS.

Motivated by this observation, we have undertaken numerical simulations of 2D turbulence at high resolution, using both normal and hyperviscosity. A dissipation proportional to Δ^4 was used for hyperviscous simulations. The goal of these simulations was twofold: first, determine which of the two scenarios is more appropriate to describe the decay, and whether there is an influence of the numerical scheme used to dissipate energy; second, determine whether there is an asymptotic transition between the value $\xi \simeq 0.7$ usually reported, and a value $\xi \approx 1$ predicted by the Kirchhoff model, or any other value. It is then necessary to consider a large number of initial vortices, so that the decay of their number occurs over several decades of time. Both viscous and hyperviscous simulations were performed with a pseudospectral code with periodic boundary conditions. We chose the resolution so that the typical size of initial vortices was outside the dissipative range. We used a 2048^2 grid for viscous simulations and a 1024^2 grid for hyperviscous ones. A random vorticity field was introduced as initial conditions. The energy spectrum corresponding to this initial field is given by $E(k) = k^{30}/(k+k_0)^{60}$. Most of the energy is concentrated at the wave number $k_0 = 100$. This corresponds to a situation with approximately 10 000 vortices randomly localized. The identification of the vortices is based on the numerical vortex selection procedure defined by McWilliams [5]. The simulations were stopped before the number of vortices became too small or their typical size too large (more than 1/50 of the box size) in order to prevent finite size effects due to the periodicity of the domain.

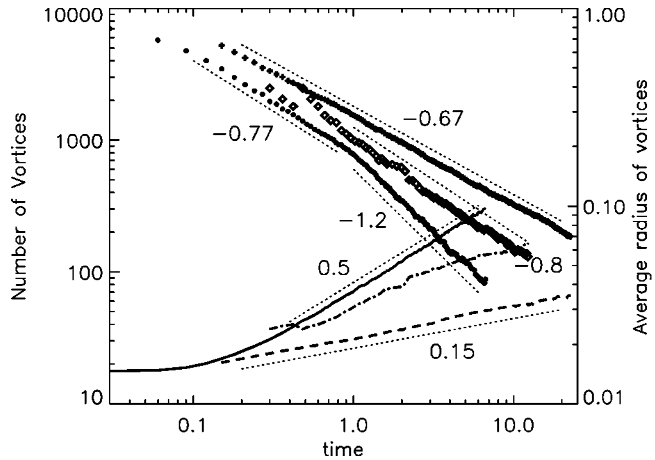

FIG. 1. Evolution of the number of vortices and their average radius for the three simulations (● and −, DNS; + and − −, HDNS; ◇ and − · −, subgrid scale model).

A good summary of the different scaling laws detected in our simulation is provided by Fig. 1. The results of the simulation with hyperviscosity mostly confirm the previous numerical simulations performed over shorter time scales. The number of vortices decays like $n \sim t^{-0.67}$ over two decades in time, while the average vortex radius and the Kurtosis increase like $a \sim t^{0.15}$ and $K \sim t^{0.30}$, respectively. The enstrophy decays steadily over the simulation like $Z \sim t^{-0.40}$, while the energy remains almost constant, especially at the later stages. Finally, the maximum of vorticity is almost conserved, decaying approximately like $t^{-0.12}$ in a first stage, even slower (like $t^{-0.06}$) in the late stage of the simulation. These last scalings are only approximate, since we did observe strong local increase of the maximum of vorticity, which we associate with strong steepening of the local vorticity profile within a few isolated vortices. This is probably an artifact of the hyperviscosity, as was previously reported. Overall, these results are compatible with the second scenario with $\xi \approx 0.7$.

When normal viscosity is adopted instead of hyperviscosity, the behavior changes dramatically. Two different regimes can be clearly distinguished: in the first one, between $t=0$ and $t \approx 1$ (or equivalently until the total number of vortices has decayed by one order of magnitude), a clean power law $n \sim t^{-0.77}$ can be observed for the total number of vortices, which is close to that obtained with hyperviscous computations. There is also a rather clean scaling law $Z \sim t^{-1.3}$ for the enstrophy within vortices decaying much steeply than in the hyperviscous computations. For any other quantities, a monotonic but nonscaling behavior is observed, with a decrease of the vorticity maximum and of the energy, and a slow increase of the average radius and of the Kurtosis. In the second regime ($t \gtrsim 1$), rather clean scaling laws for most quantities suddenly emerge, and become markedly different from the corresponding hyperviscous ones. The number of vortices decays like $n \sim t^{-1.2}$ and the average radius increases like $a \sim t^{0.50}$, resulting in an almost constant vortex area coverage and Kurtosis. This regime cannot, however, be described by the Batchelor theory, since $\xi \simeq 1.2$ instead of $\xi = 2$ and, in addition, $\omega \sim t^{-0.6}$ and $Z \sim t^{-1.3}$. To test further the observed discrepancy, we have also performed computa-

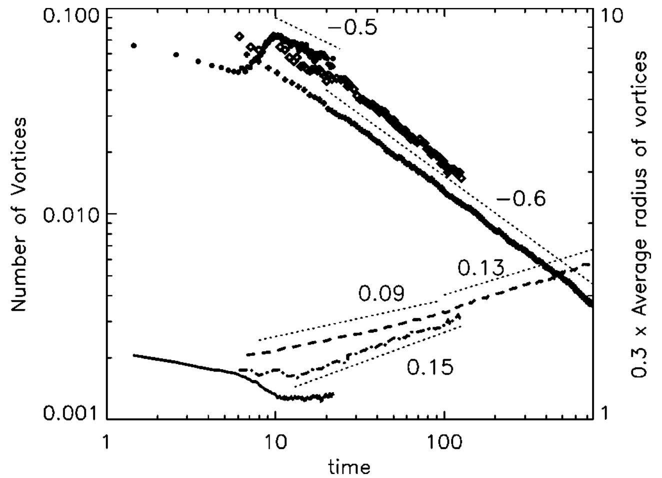

FIG. 2. Same as Fig. 1 when the number of vortices, their radius, and the time have been rescaled appropriately.

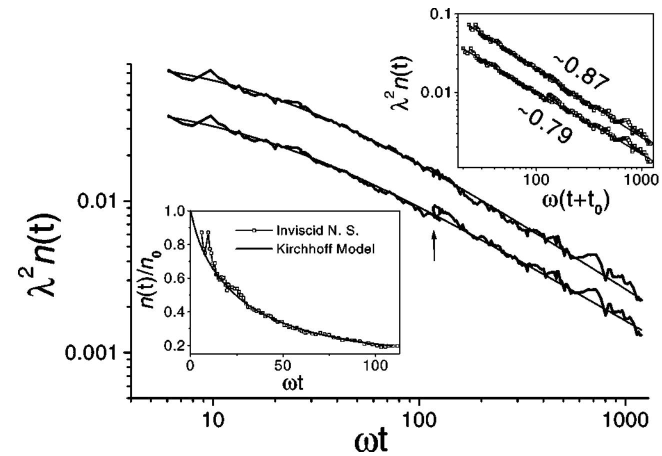

FIG. 3. Long-time evolution of the rescaled vortex density obtained using RDT simulations. In the top curve the discontinuity arising from a change of subgrid (arrow) has been smoothed out. The best fits to the functional form $n(t)=n_0/(1+t/t_0)^{\xi}$ are displayed. The same data are plotted as a function of $t+t_0$ in the top inset. The effective exponents obtained show some trends towards the asymptotic Kirchhoff model value $\xi\approx 1$ [14]. For short times, coinciding with times before the change of subgrid, the data are in agreement with the best available Kirchhoff model *polydisperse* simulations [10,13] (bottom inset).

tions using a turbulent model based on the rapid distortion theory (RDT) and described in Ref. [15]. The RDT model presumably describes the dynamics in the inviscid limit. Two different scaling regimes are observed: in the early stage of the simulation the Kurtosis remains almost constant like in the Batchelor theory but the vortex density decays with an exponent $\xi\sim 0.9$ instead of $\xi=2$. At later times, the Kurtosis increases like $K\sim t^{0.3}$, ω becomes nearly constant, and we observe scaling laws compatible with the second scenario with $\xi\sim 0.8$. During all the course of the simulation, the energy is nearly constant, like in the hyperviscous case.

Since all these results were obtained using the same initial conditions, they show that the vortex statistics is strongly influenced by the dissipative process acting at small scales. This influence has been noted before: for example, Bartello and Warn [16] noticed that dissipation affects both the self-similar properties of the one-point vorticity density, and the behavior of high order moments. Large scale dissipation (via a drag force) also influences the scaling behavior of the enstrophy cascade [17]. In the present case, the ‘‘absolute’’ scaling law exponents, obtained when the quantities are normalized with large scale quantities such as the initial energy and the size of the box, are not universal. A natural origin to this difference could be the theoretical invariants, which may or may not be conserved (or even decay in a different way) depending on the viscosity scheme used. A good example is the maximum vorticity. If this hypothesis holds, we should be able to detect a possible universality class (independent of dissipation) of scaling behaviors via an appropriate ‘‘local’’ (in time) rescaling of the quantities using the theoretical invariants (in the inviscid limit) of the scaling theory. In the Kirchhoff model, the natural unit of time is the ‘‘average’’ vorticity of a vortex defined as $\overline{\omega}=\sqrt{KZ}$ where K and Z are the kurtosis and enstrophy of the vortices ($\overline{\omega}$ is constant in the standard scaling theory). The natural unit of length is the integral scale $\lambda=\overline{\omega}/\sqrt{E}$ built with $\overline{\omega}$ and the energy E. The scaling laws obtained in these units are reported in Fig. 2. We now obtain a much better agreement between the hyperviscous and the RDT computations, where the scaling laws seem to be compatible with the scaling scenario with $\xi\sim 0.6$. The two regimes of the viscous simulation seem to collapse, apart from a small transition zone, into a single regime where $\omega/\overline{\omega}\sim(\overline{\omega t})^{0}$, $\lambda^2 n(t)\sim(\overline{\omega t})^{-0.6}$, $a/\lambda\sim(\overline{\omega t})^{0.15}$ and $Z/\overline{\omega}^2\sim K^{-1}\sim(\overline{\omega t})^{-0.33}$. We have checked that another choice of local units (e.g., a time scale defined as the square root of the enstrophy of the vortices) does not lead to such a good overlap. This is a clear indication that the Kirchhoff theory is relevant in the dynamics of decaying turbulence.

As a further check of this, we have undertaken closer comparisons with the Kirchhoff model. The scaling laws obtained in local units are reminiscent of the early stage of the Kirchhoff simulation. To test whether these scaling laws steepen into a regime in which $\xi=1$, one needs to continue the simulations over one or two more decades in time, which would represent several months of continuous integration using our numerical resources. This makes longer integrations of the viscous or hyperviscous case impossible. However, in the RDT case, we tried to use the flexibility of the subgrid scale model to move the large scale–small scale cutoff towards larger scales (following the behavior of the integral scale), thereby allowing a gain of computational time from 10 to 100. We started the simulation with the vorticity field from the DNS at $t=0.3$ when the energy is small enough at the higher wave number. The use of a coarser grid induces a small loss of enstrophy and Kurtosis via the filtering procedure used to change the cut-off scale in our model. We observed that this change of cutoff produces an artificial, cutoff dependent, new scaling regime in‘‘absolute’’ scaling coordinates but a ‘‘universal’’ scaling regime in ‘‘local’’ scaling coordinates related to vortices quantities. This universal regime is shown in Fig. 3 for $n(t)$ and clearly suggests

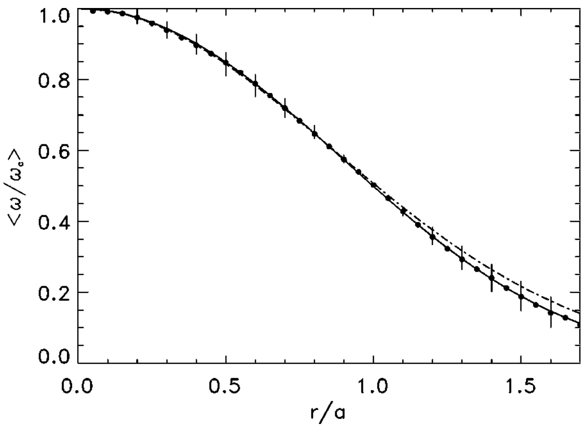

FIG. 4. Average vortex profile fitted by a function similar to a Fermi-Dirac distribution (solid line). We have also indicated the fit by a Gaussian distribution (dashed-dotted).

a local exponent ξ effectively increasing with time. The extrapolated value for ξ is $\xi \sim 0.79$ and $\xi \sim 0.87$ when the artificial discontinuity due to the change in resolution has been smoothed out (by multiplying the curve after the subgrid change by a factor ~ 0.86).

Our results therefore support the validity of a universal self-similar evolution of the vortices for inviscid, or nearly inviscid decaying turbulence. This self-similar scenario appears universal when appropriate local units are considered, and a single exponent in the range $\xi = 0.8 \sim 0.9$ is found, compatible with that of the Kirchhoff model. Finally, viscous effects tend to favor the conservation of the vortex coverage area na^2 and modify the scaling exponents without, however, leading to the Batchelor model. We have also found that the vortices present a universal profile when the vorticity is normalized by the central vorticity and the distance by the typical vortex radius defined by the condition $\omega(a) = \omega(0)/2$. This profile is represented in Fig. 4 and has been obtained by averaging over ~ 30 vortices at different times in the RDT simulation. The error bars indicate to which extent this profile can be considered as ''universal.'' As time goes on, these bars become smaller, showing a trend towards a self-similar evolution. We observe that the function $\omega = \sigma_0/(1 + \lambda e^{\alpha r^2})$ similar to a Fermi-Dirac distribution provides a very good fit to this profile, while the Gaussian distribution is less accurate (but has only one fitting parameter). In the statistical theory of 2D turbulence, the Fermi-Dirac distribution maximizes the mixing entropy introduced by Ref. [1] at fixed circulation and angular momentum (the energy constraint is not very stringent for a single merging because energy is a nonlocal quantity which can be redistributed among the other vortices). The Fermi-Dirac distribution is also the equilibrium solution of a new kinetic equation derived by Chavanis using a quasilinear theory of the 2D Euler equation [18].

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